Complex Geometry Exercises

Week 1

Exercise 1. *Prove the following theorems for holomorphic functions in n variables $f : U \to \mathbb{C}$ with $U \subseteq \mathbb{C}^n$:*

(i) Open mapping theorem

(ii) Maximum principle

(iii) (Generalised) Liouville theorem

Exercise 2. *Prove that there is a one-to-one correspondence between holomorphic germs of $\mathcal{O}_{\mathbb{C}^n,0}$ and convergent power series in $\mathbb{C}[[z_1, \ldots, z_n]]$.*

Exercise 3. *Finish the proof of the Weierstrass Division Theorem.*

Exercise 4. *Verify the following statements about analytic and holomorphic germs:*

(i) For any subset $A \subseteq \mathcal{O}_{X,x}$, $Z(A)$ is a well-defined analytic germ with $Z(A) = Z((A)_{\mathcal{O}_{X,x}})$.

(ii) For every analytic germ Z, $I(Z) = \{f \in \mathcal{O}_{X,x} \mid Z \subset Z(f)\}$ is an ideal.

(iii) If $X_1 \subset X_2$ are analytic germ, then $I(X_2) \subset I(X_1)$.

(iv) If $I_1 \subset I_2$ are ideals in $\mathcal{O}_{X,x}$, then $Z(I_2) \subset Z(I_1)$.

(v) $Z = Z(I(Z))$ and $I \subset I(Z(I))$.

(vi) $Z(I \cdot J) = Z(I) \cup Z(J)$ and $Z(I + J) = Z(I) \cap Z(J)$.

Exercise 5. *Prove the weak Nullstellensatz: If $f \in \mathcal{O}_{X,x}$ is irreducible and $g \in I(Z(f))$, then $f \mid g$.*